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OPTIMAL DISKS IN VIBRATION

FRITHIOF NIORDSON
Department of Solid Mechanics, Technical University of Denmark, Building 404,
DK-2800 Lyngby, Denmark

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Abstract—The harmonic in-plane vibrations of axisymmetrical annular disks of non-uniform thickness are analysed and the problem of finding an optimal shape is given a classical mathematical formulation with differential equations and boundary conditions. It is then shown that there is no solution to this problem satisfying all conditions. The problem is then reformulated to become well-posed. In the case of non-axisymmetrical vibrations a method using successive iterations is presented and numerical solutions are obtained. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

Optimal design of vibrating structural elements has been a subject of intense research. The first paper in this field (Niordson, 1965) was dedicated the optimal tapering of a simply supported beam. Optimal cantilever beams were discussed by Niordson and Karihaloo (1973) and circular shafts in forward precession by the same authors (1975). These were followed by several other papers on beams, plates and shells. For a review of the earlier development, see Niordson and Pedersen (1973) and Olhoff (1976). In a number of these investigations, the problem was to design an element of limited size or volume, in some cases also subject to additional restrictions, with as high a fundamental frequency as possible.

This paper deals with the optimal design of a thin axisymmetrical disk, which performs in-plane harmonic vibrations. As opposed to the case of beams and plates in bending, the equations of motion are of second, not of fourth order. A remarkable and unusual characteristic of this problem is that the design function does not appear in the condition of optimality.

Since the disk is assumed to be thin we shall neglect the motion perpendicular to the plane of the disk and we shall assume a state of plane stress. It is easily seen, that even if the fundamental frequency of a thin disk depends on its shape, it is independent of the actual thickness. Thus, if the thickness of the disk is doubled everywhere, the natural frequency remains the same. Therefore the volume of the optimal disk, if there is one, is of no significance. In the case of in-plane vibrations, the volume does, therefore, not restrict the design space and this problem differs from most other problems of optimal design of structural elements.

We investigate first the case of axisymmetrical vibrations and derive an exact and complete solution to the nonlinear differential equations. Furthermore, we show that the solution we obtain cannot satisfy the conditions of stress-free boundaries. Thus, we must conclude that, in the case of axisymmetrical vibrations, there is no optimal shape.

In the case of non-axisymmetrical vibrations the problem is more complicated since we have no explicit solution to the differential equations. However, by formulating the conditions for an initial value solution to the equations we find that the problem is overdetermined, and in general no solution satisfying the equations and the boundary conditions can be found.

However, this does not exclude the possibility to establish a numerical procedure in which we determine a sequence of shapes, each one with a higher eigenvalue than the preceding one and such that the condition of optimality is approximated with increasing accuracy. The gain over the disk of uniform thickness can then be derived and the corresponding shape determined.

2. BASIC RELATIONS

Let u and v be the in-plane displacements in radial and circumferential directions. In polar coordinates (r, ϕ) the components of the strain tensor are

$$\varepsilon_{rr} = \frac{\partial u}{\partial r} \quad \varepsilon_{\phi\phi} = \frac{u}{r} + \frac{\partial v}{r\partial \phi} \quad \varepsilon_{r\phi} = \frac{1}{2}\left(\frac{\partial u}{r\partial \phi} + \frac{\partial v}{\partial r} - \frac{v}{r}\right) \tag{1}$$

Assuming linear elastic behaviour and plane stress we can write the components of the stress tensor as

$$N_{rr} = \frac{Eh}{1-\nu^2}(\varepsilon_{rr} + \nu\varepsilon_{\phi\phi}) \quad N_{\phi\phi} = \frac{Eh}{1-\nu^2}(\varepsilon_{\phi\phi} + \nu\varepsilon_{rr}) \quad N_{r\phi} = \frac{Eh}{1+\nu}\varepsilon_{r\phi} \tag{2}$$

where E is Young's modulus, ν Poisson's ratio and $h(r)$ the thickness of the disk, assumed to be a function of r.

Assuming harmonic vibrations, the equations of motion can be written

$$\frac{\partial N_{rr}}{\partial r} + \frac{1}{r}\frac{\partial N_{r\phi}}{\partial \phi} + \frac{1}{r}(N_{rr} - N_{\phi\phi}) + \omega^2 h\gamma u = 0 \tag{3}$$

$$\frac{1}{r}\frac{\partial N_{\phi\phi}}{\partial \phi} + \frac{\partial N_{r\phi}}{\partial r} + \frac{2N_{r\phi}}{r} + \omega^2 h\gamma v = 0 \tag{4}$$

where ω is the circular frequency and γ the mass density of the material.

Introducing

$$u(r, \phi) = u(r)\cos m\phi$$

and

$$v(r, \phi) = v(r)\sin m\phi$$

where m is an (integer) number of nodal diameters, and substituting, the equations of motion can be written as the following system of two ordinary differential equations

$$u'' + u'\left(\frac{1}{r} + \frac{h'}{h}\right) - \frac{u}{r}\left(\frac{1}{r} - \nu\frac{h'}{h} + \frac{1-\nu}{2r}m^2\right) + \frac{v'm}{r}\frac{1+\nu}{2} - \frac{vm}{r}\left(\frac{3-\nu}{2r} - \nu\frac{h'}{h}\right) + \lambda u = 0 \tag{5}$$

$$v'' + v'\left(\frac{1}{r} + \frac{h'}{h}\right) - \frac{v}{r}\left(\frac{1}{r} + \frac{h'}{h} + \frac{2m^2}{r(1-\nu)}\right) - \frac{u'm}{r}\frac{1+\nu}{1-\nu} + \frac{um}{r}\left(\frac{3-\nu}{r(1-\nu)} + \frac{h'}{h}\right) + \frac{2\lambda v}{1-\nu} = 0 \tag{6}$$

where

$$\lambda = \omega^2(1-\nu^2)\frac{\gamma}{E}. \tag{7}$$

We shall assume that the disk is annular with inner radius $r = a$ and outer radius $r = b$

where $0 < a < b < \infty$ and that the boundaries are stress free, i.e.

$$N_{rr} = N_{r\phi} = 0 \tag{8}$$

at $r = a$ and $r = b$.

This is equivalent to the conditions

$$u' + \nu \frac{u+mv}{r} = 0 \tag{9}$$

$$v' - \frac{mu+v}{r} = 0 \tag{10}$$

at $r = a$ and $r = b$.

The two coupled ordinary differential eqns (5) and (6) and the four boundary conditions (9) and (10) constitute an eigenvalue problem for λ.

From the equations it is evident, that the thickness h is unessential as long as the ratio $h(r)'/h(r)$ is kept unchanged. The eigenvalue depends of course on this ratio and the aim of this paper is to find the thickness function $h(r)$, which gives the highest eigenvalue—if it exists.

Multiplying eqns (3) and (4) by $ur\,\mathrm{d}r$ and $vr\,\mathrm{d}r$, respectively, adding them together and integrating from a to b, we get an equation from which λ can be solved.

If we apply integration by parts and invoke the boundary conditions, the result will be the Rayleigh quotient

$$\lambda = \frac{\int_a^b Q(r) hr\,\mathrm{d}r}{\int_a^b (u^2+v^2) hr\,\mathrm{d}r} \tag{11}$$

where we recognize Q to be twice the strain energy density.

$$Q(r) = (u')^2 + \left(\frac{u+vm}{r}\right)^2 + 2\nu u'\left(\frac{u+vm}{r}\right) + \frac{1-\nu}{2}\left(v' - \frac{v+um}{r}\right)^2. \tag{12}$$

For the optimal disc, a variation of h would not affect the Rayleigh quotient. Let $h \to h + h_\varepsilon$ be a variation of h, where h_ε is an arbitrary function. This leads to the condition

$$\int_a^b [Q(r) - \lambda(u^2 v^2)] h_\varepsilon r\,\mathrm{d}r = 0.$$

Since this is to hold for all variations h_ε the expression in the brackets must vanish identically, and introducing Rayleigh's quotient we find the condition of optimality to be

$$Q(r) - \lambda(u^2+v^2) = 0 \tag{13}$$

which must be fulfilled for all values of r. It should be observed that the design function $h(r)$ does not appear here. This condition is more strict (the right hand side is zero, not just any constant) than in cases where there is a restriction on the volume.

The three eqns (5), (6) and (13) for the three unknown functions $u(r)$, $v(r)$ and $h(r)$ of which the last one is non-linear, together with the boundary conditions (9) and (10) constitute the eigenvalue problem for λ.

We shall now attempt to solve this problem.

3. AXISYMMETRICAL DEFORMATION

In the case of rotational symmetry ($m = 0$) the function $v(r)$ and all derivatives with respect to ϕ vanish identically. Hence eqn (6) is identically satisfied, and eqn (5) is reduced to

$$\left[h\left(u'+v\frac{u}{r}\right)\right]'+\frac{1}{r}h\left(u'+v\frac{u}{r}\right)+\lambda hu = 0 \tag{14}$$

Also, since $v(r) = 0$ the condition of optimality condenses to

$$(u')^2+\frac{2v}{r}u'u+\frac{u^2}{r^2} = \lambda u^2. \tag{15}$$

We have now two equations for the two unknown functions u and h. Together with the boundary conditions (9), which give $u'+vu/r = 0$ at $r = a$ and $r = b$, they constitute the eigenvalue problem for λ.

Equation (15) does not contain the unknown function $h(r)$ and can be solved directly. It is a second degree equation for u'/u with the solutions

$$\frac{u'}{u} = -\frac{v}{r}\pm\sqrt{\lambda-\frac{1-v^2}{r^2}} \tag{16}$$

for the solutions to be real, we must have that

$$\lambda \geqslant \frac{1-v^2}{a^2}$$

where a is the inner radius of the annual disk.

Using the notation

$$R = \lambda-\frac{1-v^2}{r^2} \tag{17}$$

we can write this equation

$$\frac{u'}{u} = -\frac{v}{r}\pm\sqrt{R}$$

which has the solutions

$$u = C_1 r^{-v}\exp\left(\pm\int_a^r\sqrt{R}\,\mathrm{d}r\right)$$

$$u = C_1 r^{-v}\exp\pm\left(r\sqrt{R}+\sqrt{1-v^2}\arctan\frac{\sqrt{1-v^2}}{r\sqrt{R}}\right). \tag{18}$$

It should be observed that since eqn (15) is non-linear, a non-trivial linear combination of the solutions (18) does not constitute a solution.

The optimal thickness function $h(r)$ can now be determined from the equation of motion (5), which by using the shorter notation R can be written

$$[hu\sqrt{R}]' + \frac{1}{r} hu\sqrt{R} \mp \lambda hu = 0.$$

Dividing through by $hu\sqrt{R}$ we get

$$(\log [hu\sqrt{R}])' + \frac{1}{r} \mp \frac{\lambda}{\sqrt{R}} = 0$$

with the integral

$$h = \frac{C_2}{ur\sqrt{R}} \exp(\pm r\sqrt{R}). \tag{19}$$

Our solution has two still unknown constants C_1 and C_2 and the yet undetermined eigenvalue λ. Since the equation of motion (14) is homogeneous, the constant C_1 will remain arbitrary, but the remaining two must be determined from the boundary conditions, which require the radial stress to vanish at the boundaries.

The radial stress according to (2) is

$$N_{rr} = \frac{Eh}{1-\nu^2}\left(u' + \nu \frac{u}{r}\right)$$

or, using eqn (19)

$$N_{rr} = C_2 \frac{E}{1-\nu^2} \frac{1}{r} \exp(\mp r\sqrt{R}).$$

Since C_2 must be non-zero, the radial stress is nowhere zero. The assumption that the condition of optimality (13) is satisfied identically, leads to this inconsistency and we must conclude that in the case of axisymmetrical deformation there is no optimal shape of the disk.

In fact, since the frequency of a thin annular ring in axisymmetrical vibrations decreases with its radius, the frequency of an annular disk increases as more material is moved and concentrated towards the inner radius, and although there is an upper bound to the frequency, determined by

$$\lambda = \frac{1-\nu^2}{a^2}$$

there is no optimal shape.

For non-axisymmetric deformations the situation is different.

4. NON-AXISYMMETRICAL DEFORMATION

Equations (5), (6) and (13) are three ordinary differential equations for the three unknown functions u, v, and h. To solve this system, we may proceed as follows. We write the equation of optimality as

$$Q \equiv \lambda(u^2+v^2)$$

and since it holds for all values of r within the disk, it also holds for the derivative with respect to r. Hence, we have

$$Q' = 2\lambda(uu'+vv') \tag{20}$$

When taking the derivative of Q the second derivatives of u and v appear. They can be eliminated, using the equations of motion (5) and (6). The resulting equation is a second degree polynomial in u and v and their first derivatives only. It is *linear* in the derivative of h and has the form

$$\frac{h'}{h}P_N(u,u',v,v')+P_T(u,u',v,v') = 0 \tag{21}$$

where both P_N and P_T are second degree polynomials in the displacements and their first derivatives. Muliplying both numerator and denominator by r^3, we have for the numerator

$$\begin{aligned} P_T = {} & u^2[2(1-\nu)+m^2(4-3\nu+\nu^2)+2\lambda\nu r^2] \\ & +v^2[2(1-\nu)+m^2(4-3\nu+\nu^2)-2\lambda r^2] \\ & +uv[m^3(2-\nu+\nu^2)+m(10-9\nu+\nu^2-2(1-\nu)\lambda r^2)] \\ & +uu'[m^2(3\nu-1)r+4(\nu-1)r+4\lambda r^3]+2(u'^2+v'^2)(1-\nu)r^2+uv'm(-8+5\nu+\nu^2)r \\ & +vu'm(-5+7\nu)r+u'v'm(1-3\nu)r^2+vv'[-4(1-\nu)r+m^2(-4+\nu+\nu^2)r+4\lambda r^3] \end{aligned} \tag{22}$$

and for the denominator

$$\begin{aligned} P_N = {} & u^2(m^2(1-\nu)+2\nu^2)r+2u'^2r^3+v^2(1-\nu+2m^2\nu^2)r+v'^2(1-\nu)r^3 \\ & +4vu'm\nu r-2vv'(1-\nu)r^2+4uu'\nu r^2+2uvm(1-\nu+2\nu^2)r-2uv'm(1-\nu)r^2. \end{aligned} \tag{23}$$

Solving for h'/h we get

$$\frac{h'}{h} = -\frac{P_T}{P_N} \tag{24}$$

and this function can now be substituted into (5) and (6) which become two non-linear differential equations for the unknown displacements. The equations also contain the unknown parameter λ, which appears in both polynomials. The solution is subject to the condition that the normal stress and the shear stress vanish at the boundaries.

Let us assume that there is a solution to the problem as defined above and let us formulate it as an initial value problem.

At the inner boundary $r = a$, we need the initial values $u(a)$, $v(a)$, $u'(a)$, and $v'(a)$.

Since the system of equations is homogeneous in the displacements, the initial value of u is arbitrary, and we can take $u(a) = 1$ without loss of generality. Furthermore, the initial values of the shear deformation at the inner boundary $v(a)$, the derivatives $u'(a)$ and $v'(a)$, which we also need, are completely determined by the three conditions of vanishing normal- and shear stress at the boundary and the optimality condition (13), provided that λ is known. Since λ is not known, we have precisely one free parameter which can be used to make the normal stresses vanish at the outer boundary, but then, of course, we cannot expect the shear stresses to vanish at the same radius and the problem is therefore over-determined. With one free parameter we cannot ensure that both stresses vanish at the outer boundary, which after all is a necessary condition.

We are therefore led to the conclusion that the condition of optimality does not hold at the boundary, since $v(a)$ cannot be determined from the condition of optimality there. The reason for this is that there is a singularity, since the denominator (23) vanishes at the boundaries, where the stresses vanish. Clearly, h'/h cannot be determined there and it is therefore not surprising that any attempt to solve the problem numerically as an initial value problem will break down.

Since the condition of optimality cannot be satisfied everywhere there is—strictly speaking—no optimum, no optimal disk and no maximum eigenvalue. In this respect our problem does not differ essentially from many other problems in optimal design, where inherent singularities lead to similar difficulties. We have so far not a well-posed problem and we must try to reformulate it.

As a starting point we shall postulate that there is an infinite set of regular shapes $h(r)$ between given inner and outer boundaries. To each member there belongs a lowest eigenvalue and a unique deviation function

$$S(r) = Q(r) - \lambda(u^2 + v^2) \tag{25}$$

which is never identically zero. As a measure of this deviation function we shall use its maximum absolute value S_{max} over the interval.

We reformulate the problem in the following way: find a sequence of shapes with higher and higher eigenvalues for which the condition of optimality is better and better approximated.

The problem that we now propose is to construct a sequence of functions $h_n(r)$ with $n = 0, 1, 2, \ldots$ such that the eigenvalues belonging to these functions increase monotonously and such that the measure S_{max} of the corresponding deviation functions S_n tend to their lower limit, which may be zero or not. Now, even if there is no maximum eigenvalue there is an upper bound for the eigenvalues of this set and we will approach this upper bound indefinably as $(S_{max})_n$ approaches zero.

In the next section we shall explore the possibilities of such an approach using numerical methods for finding a solution.

5. NUMERICAL SOLUTIONS

The central part of the numerical solution of the problem is the procedure for solving the eigenvalue problem for a given shape, i.e., for a given function $h(r)$. Although such a procedure is rather standard, we shall give a short outline of it. We use a standard Runge-Kutta method to solve the differential equations, but we shall avoid eqns (5) and (6), which contain the derivative of the thickness function and instead rely upon the equations of equilibrium (3) and (4), which do not. From a numerical point of view this is an attractive feature. Hence we use as dependent variables

$$u \quad v \quad N_{rr} \quad N_{r\phi}$$

for which we have the differential equations

$$u' = \frac{N_{rr}}{h} - \frac{\nu}{r}(u + mv) \tag{26}$$

$$v' = \frac{2N_{r\phi}}{h(1-\nu)} + \frac{1}{r}(v + mu) \tag{27}$$

$$N_{\phi\phi} = h\left(\frac{u+mv}{r} + \nu u'\right) \tag{28}$$

$$N'_{rr} = \frac{1}{r}(N_{\phi\phi} - N_{rr} - mN_{r\phi}) - \lambda hu \tag{29}$$

$$N'_{r\phi} = \frac{1}{r}(mN_{\phi\phi} - 2N_{r\phi}) - \lambda hv. \tag{30}$$

To apply the Runge-Kutta method we subdivide the radius of the disk between the inner and the outer radius in a number of (equal) parts of δr. Since the numerical error will be smaller the smaller δr is, we should choose a large number of parts.

However, since our aim is not in itself an accurate solution of the eigenvalue problem and since for obvious reasons this number has to be taken considering the available computing facility we have after some tests, covering the range from 10 to 1000 parts, taken the number of parts to be 80, which gives us the accuracy needed for our purpose. It should be noted that we use 80 parts from the inner boundary to the outer, which of course gives a closer spacing for the thinner rings.

Using the simplest Runge-Kutta method, a single step forward is determined by

$$u \to u + u'\delta r$$

$$v \to v + v'\delta r$$

$$N_{rr} \to N_{rr} + N'_{rr}\delta r$$

$$N_{r\phi} \to N_{r\phi} + N'_{r\phi}\delta r$$

and in this way we proceed step by step until we reach the outer boundary.

The equations (26)–(30) are linear and homogeneous so that any linear combination of two solutions is also a solution. We utilize this by solving two initial value problems, one with $u(a) = 1$, $v(a) = 0$ and the other with $u(a) = 0$, $v(a) = 1$ and with zero stresses $N_{rr}(a) = 0$, $N_{r\phi}(a) = 0$ at the inner boundary.

At the outer boundary we find the stresses $[N_{rr}(b)]_1$, $[N_{r\phi}(b)]_1$ for the initial conditions $u(a) = 1, v(a) = 0$ and $[N_{rr}(b)]_2$, $[N_{r\phi}(b)]_2$ for the initial conditions $u(a) = 0, v(a) = 1$. When λ is an eigenvalue, the determinant

$$[N_{rr}(b)]_1[N_{r\phi}(b)]_2 - [N_{r\phi}(b)]_1[N_{rr}(b)]_2 \tag{31}$$

must vanish. The determinant is a function of λ and we can easily determine the first zero of this function by standard methods to any desired degree of accuracy. In our procedure the zero was determined with at least 9 significant digits.

Having determined the eigenvalue and using the initial conditions

$$u(a) = 1\,; \quad v(a) = -\frac{[N_{rr}(b)]_1}{[N_{rr}(b)]_2}$$

we may solve the differential equations again using the Runge-Kutta method, and fulfil all boundary conditions. With the complete solution in hand for a given thickness function $h(r)$ we can determine the deviation from the optimality condition $S(r)$ from (25).

Let us find a sequence of successive iterations, which leads to better and better approximations of the condition of optimality, making the deviation function (25) arbitrarily small. If we succeed, the eigenvalues belonging to this sequence come arbitrarily close to the upper bound and the shape, which we determine, will come as close to the optimal shape as we wish.

Starting with a disk of uniform thickness

$$h_0(r) \equiv 1$$

and using the iteration scheme

$$h_n(r) = h_{n-1}(r) + \delta h_{n-1}(r) \tag{32}$$

where δh is the increment to the thickness function we proceed until the deviation (25) is below a prescribed limit.

As a benchmark for the approximations we shall use the maximum absolute value of the deviation function for a disk of uniform thickness $S_0(r)$ and we shall terminate the sequence of successive iterations when the maximum absolute value of the deviation function $S_n(r)$ for the disk of thickness $h_n(r)$ is less than a given fraction of the benchmark (in our numerical procedure taken to be 1/10,000).

Since h does not appear in the condition of optimality there is no obvious way to choose δh, but clearly, if the sequence is to converge, δh must converge to zero everywhere, just like S.

Let us take

$$\delta h_n(r) = cS_n(r) \tag{33}$$

where the constant c is taken to be

$$c = \frac{0.9}{\text{abs}(S_0(r))_{\text{max}}}.$$

The value 0.9 is taken as a precaution, so that the increment of the thickness in (32) does not exceed the thickness in the first or the first few iterations.

We shall devote the next section to a discussion of the numerical results. As we shall see, in most cases this choice works well.

6. RESULTS AND DISCUSSION

For the value $m = 1$ the displacements represent a rigid body motion perpendicular to the nodal diameter with zero eigenfrequency. The lowest eigenvalue $\lambda = 0$ is independent of the shape of the disk. Of course, higher eigenvalues do exist, but only when there is at least one nodal circle. In this paper we do not consider such modes. Technically speaking, for $m = 1$ the eigenvalue problem is semi-definite.

Using the method outlined in the previous section, numerical results were obtained for all values of the modal parameter from $m = 2$ to $m = 8$ and for ratios a/b between 0.1 and 0.9. The condition for convergence was that the maximum absolute value of the deviation was less than 1/100,000 of the deviation for a disk of uniform thickness and the same radii.

An immediate conclusion from the numerical work was that the sequence did not converge for arbitrarily small values of the ratio a/b. By trial and error we found that the minimum ratio depends on the number of nodal diameters as given in Fig. 1. In no case did the sequence converge for the smallest ratio tried, $a/b = 0.1$.

m	*a/b*
2	*0.1098*
3	*0.2014*
4	*0.2885*
5	*0.3368*
6	*0.3690*
7	*0.3864*
8	*0.4021*

Fig. 1. Minimum ratio a/b for convergence.

	a/b								
m	***0.1***	***0.2***	***0.3***	***0.4***	***0.5***	***0.6***	***0.7***	***0.8***	***0.9***
2	*n.c.*	*196*	*126*	*86*	*59*	*40*	*31*	*23*	*29*
3	*n.c.*	*n.c.*	*106*	*74*	*53*	*38*	*28*	*23*	*21*
4	*n.c.*	*n.c.*	*98*	*65*	*48*	*36*	*25*	*22*	*17*
5	*n.c.*	*n.c.*	*n.c.*	*59*	*43*	*33*	*24*	*21*	*17*
6	*n.c.*	*n.c.*	*n.c.*	*100*	*39*	*30*	*23*	*19*	*17*
7	*n.c.*	*n.c.*	*n.c.*	*290*	*37*	*28*	*22*	*17*	*17*
8	*n.c.*	*n.c.*	*n.c.*	*n.c.*	*66*	*26*	*20*	*15*	*16*

Fig. 2. The number of iterations used.

	a/b								
m	***0.1***	***0.2***	***0.3***	***0.4***	***0.5***	***0.6***	***0.7***	***0.8***	***0.9***
2	*n.c.*	*2.19*	*1.50*	*1.29*	*1.22*	*1.23*	*1.30*	*1.42*	*1.59*
3	*n.c.*	*n.c.*	*1.65*	*1.29*	*1.16*	*1.14*	*1.19*	*1.30*	*1.50*
4	*n.c.*	*n.c.*	*1.89*	*1.36*	*1.16*	*1.10*	*1.12*	*1.22*	*1.43*
5	*n.c.*	*n.c.*	*n.c.*	*1.48*	*1.19*	*1.09*	*1.08*	*1.16*	*1.36*
6	*n.c.*	*n.c.*	*n.c.*	*1.60*	*1.24*	*1.10*	*1.06*	*1.11*	*1.31*
7	*n.c.*	*n.c.*	*n.c.*	*1.68*	*1.31*	*1.11*	*1.06*	*1.08*	*1.26*
8	*n.c.*	*n.c.*	*n.c.*	*n.c.*	*1.37*	*1.13*	*1.05*	*1.06*	*1.22*

Fig. 3. The gain λ_{max}/λ_0.

The number of iterations used is shown in Fig. 2, where n.c. stands for "not converged". Where the number of iterations is not given, the sequence was terminated either on a "time-out" basis, the maximum number of iterations being set to 981, or due to the fact that the deviation ceased to decrease monotonously.

Defining the gain as the ratio between the eigenvalue of the optimal disk and the disk of uniform thickness, we find generally rather modest values, with the most to win at the smaller ratios. The results are summarized in Fig. 3.

In all cases without exception, the first few steps, and in particular the very first step, show considerable progress in both a diminishing deviation and an increased eigenvalue. Let us consider atypical case ($m = 5$, $a/b = 0.1$) where the sequence did not converge. The progress is shown in Fig. 4. Here the number of iterations is given on the horizontal axis in a logarithmic scale up to 981, when it was interrupted. The curve "S" shows the deviation or rest error relative to the deviation for a uniform disk. After the first iteration it has diminished to just 0.19 and henceforth it seems to approach the horizontal axis asymptotically, reaching the value 0.0052 at the last step.

The curve "L" shows the increment of the eigenvalue $\lambda - \lambda_0$ over the eigenvalue of the uniform disk. It also shows a considerable change during the first and the first few iterations, but on the other hand it does not level out and shows no sign of convergence. After a few iterations it increases almost perfectly linearly (in this logarithmic scale). It is obvious that we have not come close to the upper bound for the eigenvalue by far.

The last iteration gives the shape, seen in Fig. 5. Although it is not optimal, it has a gain of 2.43 over the uniform disk.

The progress is very different in all cases, where convergence was obtained. Let us for instance take the same mode $m = 5$, but with a larger hole, $a/b = 0.5$. Here, the sequence was interrupted after only 34 iterations, since the deviation, or rest error, was then below 1/100,000. The progress is shown in Fig. 6, which is typical for all cases that converged. It

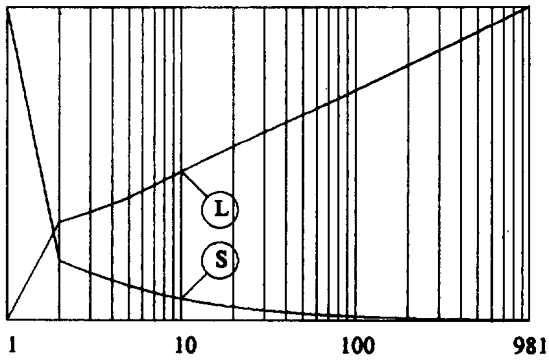

Fig. 4. Iteration progress (non-convergent) mode $m = 5$, $a/b = 0.1$.

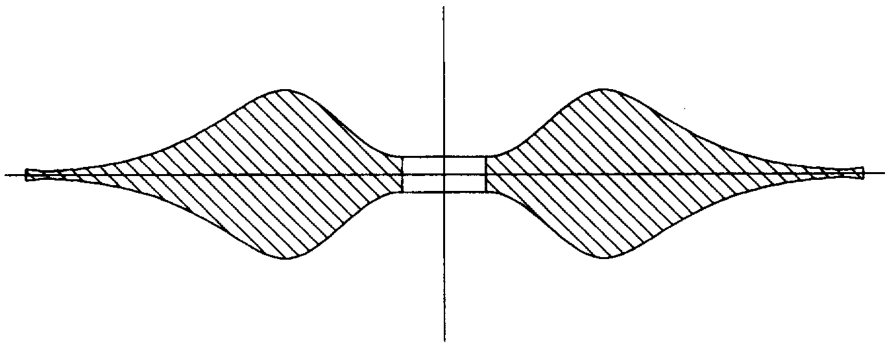

Fig. 5. Non-optimal shape mode $m = 5$, $a/b = 0.1$, gain = 2.43.

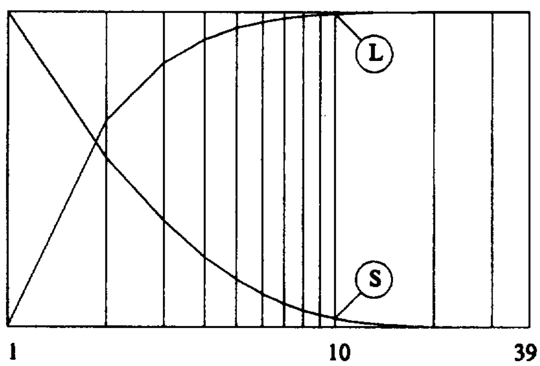

Fig. 6. Iteration progress (convergent) mode $m = 5$, $a/b = 0.5$.

is seen that the curve "L" has come close to its maximum value and the result looks convincing. The corresponding shape is shown in Fig. 7.

In all cases, where the convergence of the iterations was retarded as in the case shown in Fig. 4, the shape function $h(r)$ and the deviation function $S(r)$ were similar in shape. When this happens, the increments δh become close to proportional to h which again means that there is almost no change of shape in this step.

The shape of the optimal disks for higher ratios a/b are shown in Figs 7–9.

In the last case, Fig. 9, the disk acts very much like a curved beam. It is therefore not surprising, that in the optimal form it has found the shape of an I-beam section, since the bending of it (in the plane of the disk) has become important.

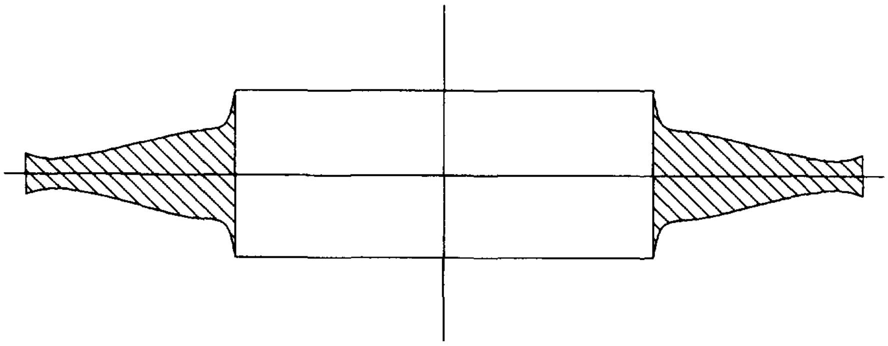

Fig. 7. Optimal shape mode $m = 5$, $a/b = 0.5$, gain $= 1.19$.

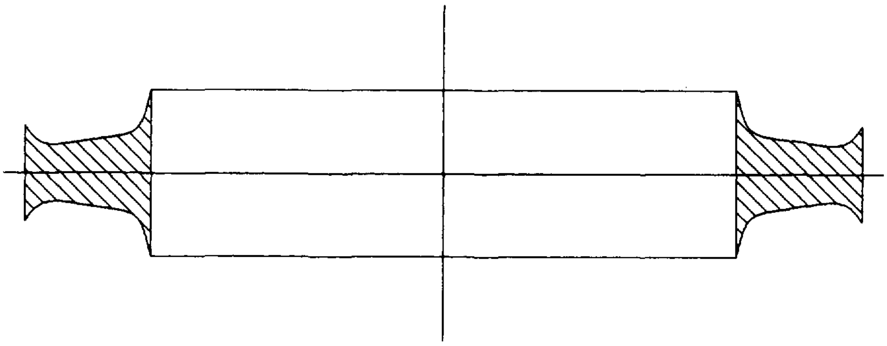

Fig. 8. Optimal shape mode $m = 5$, $a/b = 0.7$, gain $= 1.08$.

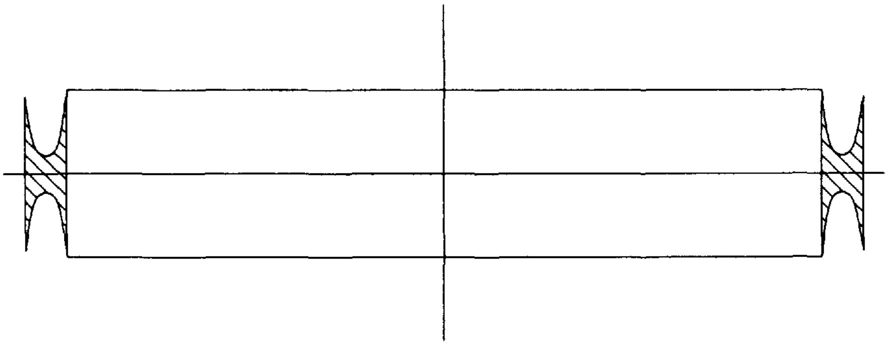

Fig. 9. Optimal shape mode $m = 5$, $a/b = 0.9$, gain 1.36.

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